2901. Evaluate $S = \displaystyle\sum_{n=1}^{\infty} \frac{2^n + 3^n}{5^n}$.

2902. A curve has equation $y = x^{\frac{1}{3}} + x^{-\frac{1}{3}}$.

(a) Find and classify any stationary points.
(b) Show that the curve has no axis intercepts.
(c) Find any asymptotes.
(d) Hence, sketch the curve.

2903. Show that there is no polynomial in $x$ which is equivalent to the following algebraic fraction:

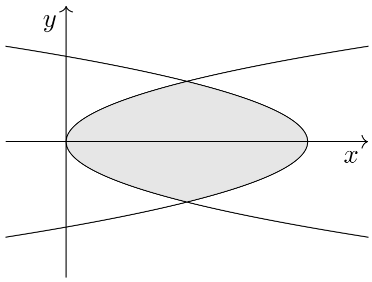
$$\frac{12x^3 - 25x^2 - x + 15}{x^3 - x^2 + x - 1}.$$

2904. Show that the area enclosed by $y = |x^3 + 8|$ and $y = 8$ is $24\left(\sqrt[3]{2} - 1\right)$.

2905. An economist is studying electricity usage in flats in a given demographic. The economist assumes that usage per day is normally distributed, with standard deviation 5.11 kWh. The suggestion is that supply changes have driven usage down from a historical value of 11.6 kWh per day. A random sample is taken, with summary statistics $n = 49$, $\sum x = 411.4$.

(a) Write down hypotheses for the test.
(b) Find the critical region at 2%.
(c) Carry out the test.
(d) Explain why the underlying assumptions of the test are unlikely, in fact, to be accurate.

2906. The curves shown are $x = y^2$ and $x = 8 - y^2$:




Find the area of the shaded region.

2907. Two monkeys of mass $m$ are holding onto opposite ends of a light, inextensible rope which runs over a smooth pulley. Initially, the two monkeys are in equilibrium. Then, one of the monkeys begins to climb. Explain what happens, with reference to Newton's laws.

2908. Determine the number of distinct roots of

$$(x^2 + x - 6)^2(x^4 + x^2 - 6) = 0.$$

2909. Show that, for $a, b \geq 1$,

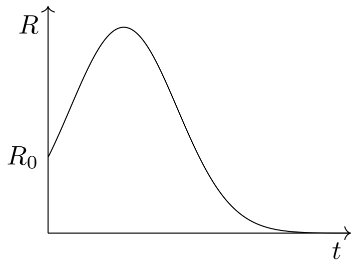
$$\int_1^{ab} \frac{1}{x}\,dx \equiv \int_1^{a} \frac{1}{x}\,dx + \int_1^{b} \frac{1}{x}\,dx.$$

2910. Give a counterexample to the following claim: "A fixed point $x_0$ of the function $\text{f}^2$ must be a fixed point of the function f."

2911. An immunologist is studying the white blood cell response to infections. The variable $R$ has units of tens of thousands of cells per millilitre of blood, and measures response over and above the baseline for a non-infected individual. The immunologist uses the model

$$\ln(2R) = 2t - t^2,$$

where $t$ is the number of days after a patient shows symptoms.




(a) Find the response at $t = 0$.
(b) Find the time at which the response falls to below the level at which symptoms showed.
(c) Show that response peaks at around 13600 white blood cells per millilitre.
(d) Determine the time at which rate of change of response is maximised.

2912. Two sequences $a_n$ and $b_n$ are both geometric.

(a) Show that $c_n = a_n b_n$ is geometric.
(b) Show that, if the sequences $a_n$ and $b_n$ have the same common ratio, then $d_n = a_n + b_n$ is also geometric.

2913. Two functions are defined, over $\mathbb{R}$, by

$$\text{S}(x) = \frac{e^x - e^{-x}}{2}, \quad \text{C}(x) = \frac{e^x + e^{-x}}{2}.$$

Prove the following identity:

$$\text{S}(a + b) \equiv \text{S}(a)\,\text{C}(b) + \text{C}(a)\,\text{S}(b).$$

2914. Show that $\displaystyle\sum_{i=-2}^{2} \frac{(-1)^i}{x + i} = 0$ has no real roots.

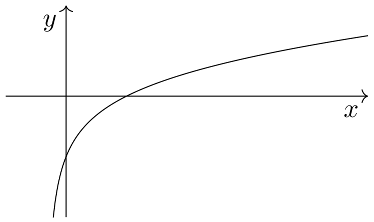
2915. Prove that, if decimal $x$ terminates after $n$ digits, then decimal $x^2$ terminates after $2n$ digits.

2916. A student writes as follows:

> "Friction acts to oppose relative motion or potential relative motion of surfaces. When a car accelerates, the potential motion is of tyres and tarmac relative to each other. Driven to rotate by the engine, the tyres would move backwards relative to the tarmac. So, friction acts forwards on the tyres and backwards on the tarmac."

Is this correct? If not, explain why not.

2917. The curve shown has equation $y = \dfrac{x-1}{\sqrt{1+3x}}$.




(a) Write down the $x$ intercept of the curve.

(b) A region is enclosed by the curve, the $x$ axis and the line $x = 21$. Show that the area of this region is 32.

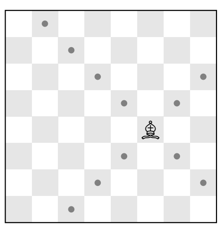
2918. Three circles $C_i$ are defined, for $i = 0, 1, 2$, by

$$\left(x - \cos\frac{2\pi i}{3}\right)^2 + \left(y - \sin\frac{2\pi i}{3}\right)^2 = \frac{1}{4}.$$

Prove that the circles are equidistant.

2919. If $y = x^2 - 1$, write the expression $2x^6 - 7x^4 + 7x^2 - 2$ in simplified terms of $y$.

2920. In chess, bishops threaten squares as shown. Find the probability that, if two bishops are placed on a chessboard at random, they threaten each other.



2921. A differential equation is given as

$$\frac{dy}{dx} + y = x + 1.$$

Show that no parabola $y = \mathrm{f}(x)$ satisfies this.

2922. Solve the inequality $81^x - 27^x + 9^x - 3^x > 0$.

2923. Show that, for $a, b > 0$, the curve $y = \log_a x$ is a stretch of $y = \log_b x$ in the $y$ direction. Give the scale factor of the stretch.

2924. Provide counterexamples to the following claims:

(a) "For a stationary point, it is sufficient that the first derivative changes sign."

(b) "For a point of inflection, it is sufficient that the second derivative is zero."

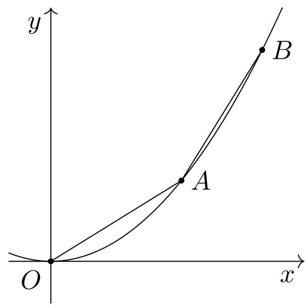
2925. The following integrals all have value zero. In each case, sketch $y = I$, where $I$ is the integrand. Mark the regions whose signed areas cancel to produce zero + or − according to their contribution to the integral.

(a) $\displaystyle\int_{-1}^{1} x^3 - x\, dx$,

(b) $\displaystyle\int_{0}^{e} \ln x\, dx$,

(c) $\displaystyle\int_{0}^{2\pi} \sin x - \cos x\, dx$.

2926. Points $A : (a, a^2)$ and $B : (1, 1)$ are drawn on the parabola $y = x^2$, with $0 < a < 1$. Chords $OA$ and $AB$ are the same length:




Determine the exact value of $a$.

2927. Show that no line of the form $y = 2x + k$ is tangent to $\sqrt{x+y} - \sqrt{x-y} = 1$.

2928. An object rests on a rough plane, which is inclined at an angle $\theta$ to the horizontal. Prove that the angle of friction, i.e. the steepest angle for which the object will not slide, is given by $\theta = \arctan \mu$, where $\mu$ is the coefficient of friction.

2929. Show that there are 1680 ways of partitioning the set of integers $\{1, 2, ..., 9\}$ into three sets of three.

2930. From the formulae $s = ut + \frac{1}{2}at^2$ and $s = vt - \frac{1}{2}at^2$, derive the formula $v^2 = u^2 + 2as$.

2931. Simplify $(a - b)\displaystyle\sum_{k=0}^{n-1} a^{n-1-k} b^k$.

2932. This question concerns the definite integral

$$I = \int_1^2 \sin(\ln x)\,dx.$$

(a) Find the equation of the tangent line $T$ to $y = \sin(\ln x)$ at $x = 1$.

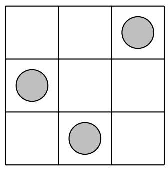
(b) Show that the curve is concave for $x \in [1, 2]$.

(c) Hence, show that $T$ lies above $y = \sin(\ln x)$ over the domain $(1, 2]$.

(d) Hence, show that $I < \frac{1}{2}$.

2933. Prove the identity $\sec 2\theta \equiv \dfrac{1 + \tan^2\theta}{1 - \tan^2\theta}$.

2934. On a three-by-three grid, three identical counters are placed at random, on distinct squares.



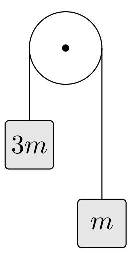
Find the probability that no two counters occupy the same row or column.

2935. A student is differentiating by the product rule. His working is as follows:

$$y = e^x \cos(3x + 1)$$
$$\Longrightarrow \frac{dy}{dx} = e^x \cos(3x + 1) - e^x \sin(3x + 1).$$

Explain the error and correct it.

2936. Two small blocks, with masses $m$ and $3m$ kg, are connected by a light, inextensible cord, which is passed over a rough pulley. The friction in the pulley causes the tensions either side of it to differ by $\frac{1}{4}R$ Newtons, where $R$ is the total downwards force on the pulley.




(a) Show that the tensions are in the ratio $5 : 3$.

(b) Hence, show that the system accelerates at $\frac{2}{7}g$.

2937. Prove that, in a Pythagorean triple $(a, b, c)$, if $c$ is even, then so are $a$ and $b$.

2938. Find all $(x, y)$ values which simultaneously satisfy

$$y = xe^x,$$
$$y = x(6e^{-x} - 1).$$

2939. A function is defined, over a suitable domain, by

$$f(x) = \frac{6x - 2}{x - 2}.$$

(a) Show that the best linear approximation to $f(x)$ at $x = 0$ is $f_1(x) = 1 - \frac{5}{2}x$.

(b) Explain how you know, without any algebraic manipulation, that the equation $f(x) = f_1(x)$ has a repeated root.

2940. Determine, in terms of the positive constant $a$, the area of the region of the $(x, y)$ plane whose points simultaneously satisfy the following inequalities:

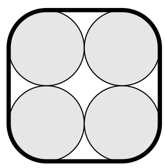
$$(x + y)^2 < a,$$
$$(x - y)^2 < a.$$

2941. Five beads are threaded onto a circular bracelet. Each of the beads is either black or white. Show that there are eight possible configurations.

2942. A curve is defined implicitly by $x^2 - \sqrt{x + y} = 1$. Show that

$$\frac{dy}{dx} = 4x\sqrt{x + y} - 1.$$

2943. In a lumber yard, four logs, each of radius $r$ and weight $W$, are bound together, in equilibrium, with a loop of smooth light rope, as shown below. The logs are modelled as cylinders, whose axes of symmetry are horizontal. The tension in the rope is $2W$ throughout.



(a) Determine the total length of the loop of rope, giving an exact answer in terms of $r$.

(b) Explain why there can be no friction acting between the two upper logs.

(c) Determine, in terms of $W$, the reaction force between upper and lower logs.

2944. A point is chosen at random on the interior of the unit circle $x^2 + y^2 = 1$. Find the probability that this point satisfies

$$y > \left| x\sqrt{5 - 2\sqrt{5}} \right|.$$

2945. Prove that no four distinct points on a cubic graph $y = f(x)$ lie on the same parabola $y = g(x)$.

2946. State, with a reason, a general formula for

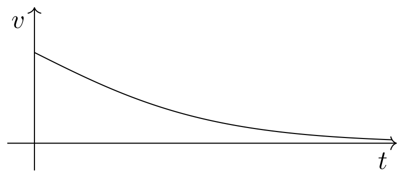
$$\int \mathrm{f}'(x)e^{\mathrm{f}(x)}\,dx.$$

2947. Determine the shortest distance between the graphs $y = |2x - 6|$ and $x^2 - 2x + y^2 + 4y + 4 = 0$.

2948. The velocity of a particle, in $\text{ms}^{-1}$, is modelled, for $t \in [0, \infty)$, by

$$v = \frac{1}{e^t + 1}.$$

A velocity-time graph is shown below:




The total displacement, in the limit as $t \to \infty$, is denoted $S$ metres.

(a) Using the substitution $u = e^t + 1$, show that

$$S = \int_2^\infty \frac{1}{u(u-1)}\,du.$$

(b) Using partial fractions, show that the model predicts a long-term displacement of $S = \ln 2$ metres.

2949. Two dice, one with $m$ sides and one with $n$, where $m < n$, are rolled together. Find, in terms of $m$ and $n$, the probability that the scores are the same.

2950. A student says: "If there is a sign change in $\mathrm{f}(x)$ between $x = a$ and $x = b$, then there is a root of $\mathrm{f}(x) = 0$ in the interval $(a, b)$." Either prove or disprove this statement.

2951. Solve $\sin^3 x + \cos^2 x + 1 = 0$ for $x \in [0, 2\pi)$.

2952. Prove, by contradiction, that $\log_3 5$ is irrational.

2953. It is given that $y = \mathrm{f}(x)$ has rotational symmetry around the origin, and also that

$$\int_0^1 \mathrm{f}(x)\,dx = k.$$

Find the following integrals, in terms of $k$:

(a) $\displaystyle\int_{-1}^{1} \mathrm{f}(x)\,dx$,

(b) $\displaystyle\int_0^{\frac{1}{2}} \mathrm{f}(2x) - \mathrm{f}(-2x)\,dx$

2954. Variables $x_1, x_2, x_3, ...$ are related as follows:

$$\frac{dx_r}{dx_{r+1}} = 2, \quad \text{for } r = 1, 2, 3, ...$$

Show that $x_n = 2^{n-1}x_1 + c$.

2955. In a court case, a witness has identified a distant car in city $C$ as a saloon. The prosecutor tested people of the same demographic, asking them to identify similar vehicles as saloon or hatchback. In 96% of experiments, they identified correctly. The prosecutor claims, based on this, that there is a 1 in 25 chance of the car having been a hatchback.

(a) Explain why, if the numbers of hatchbacks and saloons in city $C$ are known to be unequal, then the probability will not be 1 in 25.

(b) If there are three hatchbacks to every saloon in city $C$, determine the probability that a car identified as a saloon is, in fact, a hatchback.

2956. Determine the maximum value, for $x \in (0, \infty)$, of the following expression:

$$\frac{1}{1 - \ln x + (\ln x)^2}.$$

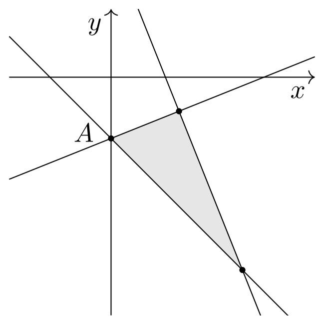
2957. A normal is drawn to the curve $y = x^3 - x + c$ at its point of inflection. Show that the normal and the curve enclose two regions, each of area 1 square unit.

2958. A quadratic graph $y = \mathrm{f}(x)$, where $\mathrm{f}(x)$ has leading coefficient $-1$, has its vertex at

$$\left(\frac{a+b}{2}, \frac{(a-b)^2}{4}\right).$$

Show that the quadratic has roots at $a$ and $b$.

2959. Triangle $T$ is formed by the lines $5x + 5y + 9 = 0$, $2x - 5y = 9$ and $5x + 2y = k$. It has area $\frac{406}{75}$.




(a) Show that $T$ is right-angled.

(b) Find the coordinates of vertex $A$.

(c) Show that the interior angle at vertex $A$ has value $45° + \arctan \frac{2}{5}$.

(d) Hence, determine the ratio of the lengths of the perpendicular sides.

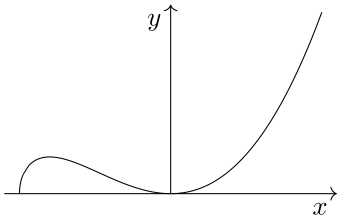
(e) Hence, find the value of $k$.

2960. Show that there are 210 possible arrangements of the letters of the word PIZZAZZ.

2961. Sketch $y = x^8 - x^4$.

2962. Non-constant sequences $a_n$ and $b_n$ are geometric and arithmetic respectively. Their terms satisfy $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_5$. Find the common ratio $r$.

2963. Determine whether $y = x^2\sqrt{x+2}$ could be the equation generating the following graph:




2964. Show that there are 48 different ways of arranging the letters of the word EARTH so that the vowels are next to each other.

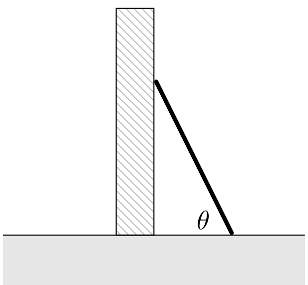
2965. In this question, do not use a calculator.

Solve $3\sqrt{x} + \dfrac{2}{\sqrt{x}} = 7$.

2966. Show that $X \sim B(6, 0.27)$ is a counterexample to the following claim: "The mode of $B(n, p)$ is the integer closest to the mean $np$."

2967. By factorising as a product of two quadratics, show that $x^4 + 2x^3 + 15x^2 + 14x + 33 = 0$ has no real roots.

2968. A ladder stands in equilibrium on rough ground, coefficient $\mu$, resting at an angle $\theta$ to the horizontal against a smooth vertical wall.




(a) Show that $\mu \geq \frac{1}{2}\cot\theta$.

(b) Comment on this value as $\theta$ approaches zero.

2969. Write down the integral of $\cos x \sin^2 x$.

2970. Given that $x = \log_2 3$ is a root, determine the full exact solution of the following equation

$$2^{x+1} + 4^{x+1} = 8^x + 15.$$

2971. The standard trapezium rule uses right-angled trapezia of equal width to estimate area. Another trapezium rule (non-standard) uses right-angled trapezia of *equal chord length*. In this question, the rules are compared by approximating

$$\int_0^1 x^2\,dx = \tfrac{1}{3}.$$

In each case, two strips are used.

(a) Show that the standard trapezium rule gives an approximation of $\frac{3}{8}$.

(b) Show that the alternative trapezium rule gives an approximation that is less good.

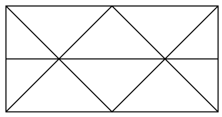
2972. A particle has position given by

$$x = \sin t, \quad y = 2\cos 2t.$$

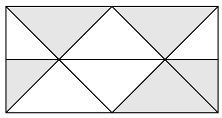
Determine the first time $t > 0$ at which the particle is moving in the direction of the vector $\mathbf{i} + 4\mathbf{j}$.

2973. Using calculus, or otherwise, prove that the curve $y = x^6 - x^2 + 1$ does not intercept the $x$ axis.

2974. A schematic map of ten regions is given as



Five of the ten regions of this schematic map are shaded, at random. Below is one possible outcome.



Find the probability that no two shaded regions share a border.

2975. This question is about the iteration $x_{n+1} = \ln x_n$.

(a) Sketch $y = \ln x$.

(b) Find the equation of the tangent to the curve that is parallel to the line $y = x$.

(c) Hence, prove that the iteration $x_{n+1} = \ln x_n$ has no fixed points.

2976. The interior angles of a hexagon form an AP. Give, in radians, the set of possible values for the third largest angle.

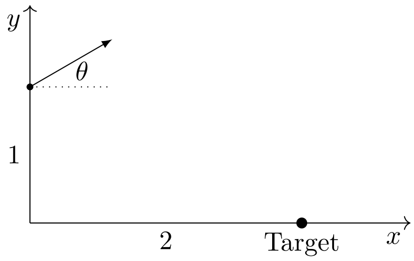
2977. Two samples of bivariate data $(x, y)$ each have a correlation coefficient $r \approx 0$. When combined, however, the samples have $r \approx 0.5$. Explain, with reference to a sketched scatter diagram, how this is possible.

2978. Find $1 + 3 + 9 + 27 + ... + 3^k$ in terms of $k \in \mathbb{N}$.

2979. Sketch $y = \sqrt[4]{x - 3}$

2980. Take $g = 10$ in this question.

A projectile is fired from a height of 1 metre above horizontal ground, in an attempt to hit a target at ground level, 2 metres away horizontally. Its initial velocity is 3 ms$^{-1}$ at an angle $\theta$ above the horizontal.




(a) Show that, if the projectile is to hit the target, then $\theta$ must satisfy

$$20 \sec^2 \theta - 18 \tan \theta - 9 = 0.$$

(b) Hence, show that a projectile launched in such a manner cannot reach the target.

2981. The edges of a hexagon are randomly coloured red or blue. Find the probability that the result is four contiguous edges of one colour and two contiguous edges of the other.

2982. Show that, if $|x - 1|$ is small, then $\ln x \approx x - 1$.

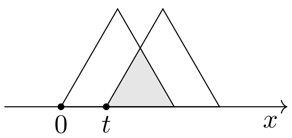
2983. The formula $x = x_0 + v_0 t + \frac{1}{2} a t^2$ describes constant acceleration, where acceleration is the second time derivative of position.

This question is about motion with constant *jerk*, where jerk is the third time derivative of position.

Find a formula for position $x$ if an object has

- initial position $x_0$,
- initial velocity $v_0$,
- initial acceleration $a_0$,
- constant jerk $j$.

2984. Two equilateral triangles with unit side length are placed with their bases on the $x$ axis, as shown. One triangle moves rightwards, with its position $x$ parametrised by time, so that $x = t$.




Draw a graph of the area $A$ of the shaded region against $t$, marking in any relevant coordinates.

2985. Write the following in terms of $\ln x$:

(a) $\ln \frac{1}{x}$,

(b) $\log_{e^2} x$,

(c) $\log_{\frac{1}{e}} x$.

2986. A chord is drawn to the curve $y = x^2 - x^3$ at $x = \pm 1$. Show that this chord is a tangent.

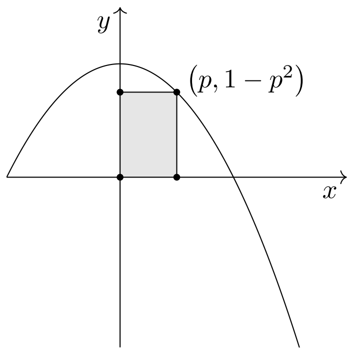
2987. Eliminate $t$ to find the Cartesian equation of the following parametrically defined curve.

$$x = 2t - t^2,$$
$$y = 2t + t^2.$$

2988. Prove that, if $T_n = \frac{1}{2} n(n + 1)$, then, for $a, b \in \mathbb{N}$,

$$T_{ab} = T_a T_b + T_{a-1} T_{b-1}.$$

2989. A rectangle has vertices at $(0, 0)$, $(p, 0)$, $(0, 1 - p^2)$ and $(p, 1 - p^2)$, for $p \in [0, \infty)$. The area of the rectangle is defined as $A$.




(a) Show that $A$ is unbounded as $p \to \infty$.

(b) Show that $A$ is zero for two values of $p$.

(c) Find the maximum value of $A$.

(d) Sketch a graph of $A$ against $p$.

2990. Explain why the Newton-Raphson method will fail to find a root of $x^2 - e^{x^2} + 2 = 0$, if the starting value is $x_0 = 0$.

2991. The exponential function can be approximated, for values close to $x = 0$, by a quadratic function $\text{g}(x) = a_0 + a_1 x + a_2 x^2$.

(a) Show that, to match $x \mapsto e^x$, the function g must have $\text{g}(0) = \text{g}'(0) = \text{g}''(0) = 1$.

(b) Hence, find the coefficients $a_0, a_1, a_2$.

(c) Sketch $y = e^x$ and $y = \text{g}(x)$ together.

2992. A right-angled triangle has area 60 and perimeter 40. Determine its side lengths.

2993. Distinct cubic functions f and g have $f(k) = g(k)$ and $f'(k) = g'(k)$ for some constant $k \in \mathbb{R}$.

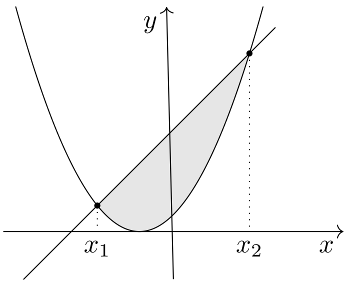
Prove that the equation $f(x) = g(x)$ has at most two distinct real roots.

2994. A curve is given as $y = \dfrac{5 - 2x - 4x^2}{5 - x}$.

(a) Write down the vertical asymptote.

(b) Find the equation of the oblique asymptote.

2995. The graph shows the parabola $y = (x + 1/2)^2$ and the line $y = x + c^2 + 1/4$, where $c > 0$. The area of the shaded region is denoted $A$.




(a) Show that $x_2 - x_1 = 2c$.

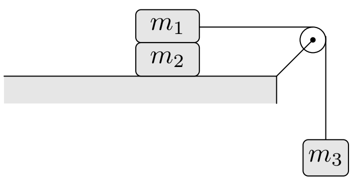
(b) Show that the vertical distance between the curve and the line is maximised at the $y$ axis.

(c) Hence, show that $A < 2c^3$.

2996. In a trick, a magician uses two coins; one is fair, the other is biased. The biased coin shows heads 75% of the time. An audience member chooses a coin at random, and flips it. The coin shows heads. Find the probability that the audience member chose the biased coin.

2997. Simplify $\log_{p^2}(p^n) \times \log_{p^n} p$.

2998. A pulley system is set up on a table as depicted. The pulley is light and smooth, and the string is light and inextensible. The coefficient of friction at the upper surface of the lower block is $\mu_1$ and at its lower surface is $\mu_2$.




Find, in terms of $m_1, m_2, m_3$, the values of $\mu_1, \mu_2$ for which both contacts are in limiting equilibrium.

2999. By first writing the integrand in partial fractions, show that

$$\int \frac{3x+3}{x^2+3x}\,dx = \ln\left|x^3 + 6x^2 + 9x\right| + c.$$

3000. Find any axis intercepts and stationary points of the following curve, and hence sketch it for $x \geq 0$:

$$y = \frac{e^x}{x + e^x}.$$

——— End of Volume III ———